

PLURISUBHARMONIC DEFINING FUNCTIONS, GOOD VECTOR FIELDS, AND
EXACTNESS OF A CERTAIN ONE FORM^{1,2}

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Abstract. We show that the approaches to global regularity of the $\bar{\partial}$ -Neumann problem via the methods listed in the title are equivalent when the conditions involved are suitably modified. These modified conditions are also equivalent to one that is relevant in the context of Stein neighborhood bases and Mergelyan type approximation.

This paper is concerned with the relationship between some conditions, listed in the title, that are known to imply global regularity of the $\bar{\partial}$ -Neumann problem. While these conditions are clearly related, they are known not to be equivalent. We show that under certain natural modifications they become equivalent. Interestingly, these (modified) conditions are also equivalent to one that is relevant in a somewhat different context: there should exist *conjugate* normal fields that are (approximately) holomorphic in weakly pseudoconvex directions. Under favorable circumstances, this leads (in addition to global regularity) to the existence of Stein neighborhood bases and to Mergelyan type approximation theorems ([1], [11]; see Remark 2 below).

While these results are of interest from the general point of view of understanding global regularity of the $\bar{\partial}$ -Neumann problem, concrete motivation came from our work in [15], where we observed that in the special situation considered there, the construction of the vector fields having good commutation properties with $\bar{\partial}$ is equivalent to the construction of a defining function plurisubharmonic at the infinite type points of the boundary of the domain (see [15], Remark 5; see also Remark 2 below for further details).

For background on the $\bar{\partial}$ -Neumann problem, we refer the reader to [10], [5], [7]. Denote by ρ a smooth (C^∞) defining function for Ω . In [4], Boas and the first author formulated a (necessarily) technical condition in terms of a family of vector fields that have good commutation properties with $\bar{\partial}$. There should exist a positive constant $C > 0$ such that for every $\varepsilon > 0$, there exists a vector field X_ε of type (1,0) whose coefficients are smooth in a neighborhood U_ε in \mathbb{C}^n of the set

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K of boundary points of Ω of infinite type and such that

$$C^{-1} < X_\varepsilon \rho < C \quad \text{on } K, \quad (1)$$

and

$$|\partial \rho([X_\varepsilon, \partial / \partial \bar{z}_j])| < \varepsilon \quad \text{on } K, \quad 1 \leq j \leq n. \quad (2)$$

We will say for short that such a family of vector fields is *transverse to $b\Omega$ ((1)) and commutes approximately with $\bar{\partial}$ ((2)) at points of K* . It was shown in [4] that if $b\Omega$ admits such a family, then the $\bar{\partial}$ -Neumann operators N_q and the Bergman projections P_q , $0 \leq q \leq n$, are continuous in Sobolev norms for $s \geq 0$. (Here, we consider the standard \mathcal{L}^2 -Sobolev spaces.) A detailed discussion of the “vector field method” may also be found in [5]. We note that the condition we have formulated here is slightly more stringent than what is required in [4], in that $X_\varepsilon \rho$ is required to be real on K (but not necessarily near K). But this is satisfied in [4] and the other situations where the vector field method works to establish regularity in W^s for all $s \geq 0$, see [3], [4] and the recent [14] and [15]. It is not clear at present how much more stringent this condition actually is.

It was already noted in [3] that in (2), it suffices to consider commutators with vector fields in Levi null directions: the commutators in the remaining directions can be adjusted by modifying X_ε in complex tangential directions (see [3], proof of the lemma for details). On the other hand, complex tangential components do not contribute to the normal (1,0)-component of commutators with fields in Levi null directions (this is a consequence of pseudoconvexity). Accordingly, we are led to consider fields which are (real) multiples of the normal to the boundary.

For such a field, it is immaterial whether we consider the normal (1,0)-component of commutators with type (1,0) or type (0,1) derivatives (in complex tangential directions), equivalently, whether we consider the normal or its conjugate. Doing the latter has the advantage that complex tangential components of type (0,1) derivatives in Levi null directions are automatically zero (details will be provided below). We are thus led to the following notion. We say that $b\Omega$ admits a *family of conjugate normals which are approximately holomorphic in weakly pseudoconvex directions* if there is a constant $C > 0$ such that for all $\varepsilon > 0$, there exists a vector field $N_\varepsilon = e^{h_\varepsilon} \sum_{j=1}^N \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial z_j}$ such that

$$-C < h_\varepsilon < C, \quad (3)$$

and

$$\bar{Y}(\bar{N}_\varepsilon)(z) = O(\varepsilon), \quad Y(z) \in N(z), \quad z \in K, \quad (4)$$

where $N(z)$ denotes the null space of the Levi form at z , Y is a (local) section of $T^{1,0}(b\Omega)$ of unit length, and \bar{Y} acts componentwise on \bar{N}_ε .

When computing (normal components of) commutators, as in (2), there is a 1-form that arises naturally. Denote by η a purely imaginary, non-vanishing smooth one-form on $b\Omega$ that is zero on the complex tangent space and its conjugate. Let T be the purely imaginary tangential vector field on $b\Omega$ that is orthogonal to the complex tangent space and its conjugate (in the metric induced by \mathbb{C}^n) and that satisfies $\eta(T) \equiv 1$ on $b\Omega$. Set $\alpha := -\mathcal{L}_T\eta$, that is, α is minus the Lie derivative of η in the direction of T . The form α was introduced into the literature by D'Angelo [8], [9]. The relevance of α in the context of global regularity was recognized in [4]. A crucial property of α is the following closedness property (this property hinges on pseudoconvexity): the differential $d\alpha$ restricted (pointwise) to the null space of the Levi form vanishes. We refer the reader to [4], section 2, for this and other properties of α (see also [5], pp. 97–98, [15]).

We say that α is *approximately exact on the null space of the Levi form* if there exists a constant $C > 0$ such that for all $\varepsilon > 0$ there exists a smooth real-valued function h_ε in a neighborhood U_ε (in $b\Omega$) of K such that on K

$$1/C \leq h_\varepsilon \leq C, \tag{5}$$

and

$$dh_\varepsilon/N(p) = \alpha/N(p) + O(\varepsilon). \tag{6}$$

Here, $O(\varepsilon)$ denotes a 1-form that satisfies $|O(\varepsilon)(X)| \leq \text{const. } \varepsilon|X|$. Note that although α depends on the choice of η , whether or not α is approximately exact on the null space of the Levi form does not: direct computation shows that if $\tilde{\eta} = e^g\eta$ for some smooth function g , then the corresponding form $\tilde{\alpha}$ differs from α , on complex tangent vectors, by the differential of g .

The remaining condition alluded to in the title is as follows. We say that Ω admits a defining function that is plurisubharmonic at the boundary ([3]), if there exists some smooth defining function ρ whose complex Hessian is positive semi-definite at all boundary points. This condition is slightly more stringent than pseudoconvexity, which requires positive definiteness only on the complex tangent space (rather than in all directions). It was shown in [3] that if Ω admits a defining function that is plurisubharmonic at the boundary (near the points of infinite type is sufficient), then $b\Omega$ admits a family of vector fields transversal to the boundary and commuting approximately with $\bar{\partial}$ (and the Bergman projection and the $\bar{\partial}$ -Neumann operator are regular in Sobolev norms). It was pointed out in [4], Remark 3, that the existence of such a family of vector fields is actually a weaker property than the existence of a defining function that is plurisubharmonic at the boundary: examples may be obtained by considering domains that have as suitable lower dimensional sections domains not admitting (even) a (local) plurisubharmonic defining function. (See [4], Remark 3 for references.) However, the proof in [3] does not use the full force of the positive definiteness of the complex Hessian L_ρ at boundary points; it suffices to have this on the span of the null space of the

Levi form and the complex normal to the boundary. On this span, however, it amounts to the same to assume that the complex Hessian is zero. Indeed, if $\tilde{\rho} = g\rho$, then the complex Hessian $L_{\tilde{\rho}}$ of $\tilde{\rho}$ satisfies (on the boundary) $L_{\tilde{\rho}}(X, \bar{Y})(P) = g(P)L_{\rho}(X, \bar{Y})(P) + X\rho(P)\bar{Y}g(P) + \bar{Y}\rho(P)Xg(P)$. Now if $Y(P) \in N(P)$, then the semidefiniteness of L_{ρ} on the span of $N(P)$ and the complex normal at P implies that $L_{\rho}(X, \bar{Y})(P) = 0$ for X in this span (it is only for this conclusion that plurisubharmonicity of ρ was used in [3]). Consequently, if we choose $g \equiv 1$ on $b\Omega$, then also $L_{\tilde{\rho}}(X, \bar{Y})(P) = 0$. Extending g from the boundary so that the (real) normal derivative equals minus $L_{\rho}(Z, \bar{Z})$, where Z is the complex normal (normalized so that $Z\rho \equiv 1$ on $b\Omega$) gives that in addition also $L_{\tilde{\rho}}(Z, \bar{Z}) = 0$. Thus $L_{\tilde{\rho}}(P)$ is zero on the span of $N(P)$ and the complex normal. Finally, as with the other conditions, we only need this condition to be satisfied approximately. Accordingly, we say that Ω admits a family of essentially pluriharmonic defining functions if there exists a constant $C > 0$ such that for all $\varepsilon > 0$ there exists a (C^∞) defining function ρ_ε for Ω satisfying

$$1/C \leq |\nabla \rho_\varepsilon| \leq C, \quad (7)$$

and

$$\left| \sum_{j,k} \frac{\partial^2 \rho_\varepsilon(P)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \right| \leq O(\varepsilon) |w|^2 \quad \forall w \in \text{span}_{\mathbb{C}}\{N(P), L_n(P)\} \quad (8)$$

for all boundary points P in K . Here $\text{span}_{\mathbb{C}}$ denotes the linear span over \mathbb{C} , and L_n is the complex normal (see below for the normalization we will use).

Theorem. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^n . The following are equivalent:*

- (i) *Ω admits a family of essentially pluriharmonic defining functions*
- (ii) *Ω admits a family of conjugate normals which are approximately holomorphic in weakly pseudoconvex directions*
- (iii) *Ω admits a transversal family of vector fields which commute approximately with $\bar{\partial}$*
- (iv) *the form α associated to some choice of η (hence to any choice) is approximately exact on the null space of the Levi form.*

Remark 1. We emphasize again that condition (i) in the theorem is indeed a generalization of the notion of plurisubharmonic defining function, as explained in the discussion preceding the definition of a family of essentially pluriharmonic defining functions.

We begin with the equivalence of (iii) and (iv). That (iv) implies (iii) was observed by Boas and the first author in [5] (see the discussion at the end of section 6, the ideas are from [4], [3]);

we will recall the main points here. Also, a version of (iii) \Rightarrow (iv), in the case when the infinite type points are contained in submanifolds of the boundary of a certain kind, was pointed out in [4], Remark 5.

It will be convenient to choose $\eta := \partial\rho - \bar{\partial}\rho$, where ρ is a smooth defining function for Ω , and $T := L_n - \bar{L}_n$, where $L_n := \frac{2}{|\nabla\rho|^2} \sum_{i=1}^n \frac{\partial\rho}{\partial\bar{z}_i} \frac{\partial}{\partial z_i}$. If Y is a local section of $T^{1,0}(b\Omega)$, then the definition of the Lie derivative, the fact that $\eta(\bar{Y}) \equiv 0$, and the fact that $(\partial\rho + \bar{\partial}\rho)([L_n - \bar{L}_n, \bar{Y}]) = 0$ give

$$\alpha(\bar{Y}) = 2\partial\rho([L_n, \bar{Y}]) \quad (9)$$

(compare [9], p. 92, [4], p. 231). If L_1, \dots, L_{n-1} denote local sections of $T^{1,0}(b\Omega)$ that span $T^{1,0}(b\Omega)$ (locally), then the fields X_ε in (iii) can be written (locally) as

$$X_\varepsilon = e^{h_\varepsilon} L_n + \sum_{j=1}^{n-1} a_j^\varepsilon L_j, \quad (10)$$

with smooth functions a_j^ε ($1 \leq j \leq n-1$) and h_ε . Computing commutators with \bar{L}_k , $1 \leq k \leq n-1$, and taking normal (1,0)-components gives (keep in mind that with the normalization above, $(\partial\rho - \bar{\partial}\rho)(L_n - \bar{L}_n) = 1$, but $\partial\rho(L_n) = \frac{1}{2}$)

$$\begin{aligned} \partial\rho([X_\varepsilon, \bar{L}_k]) &= \left(-\frac{1}{2} \bar{L}_k h_\varepsilon + \partial\rho([L_n, \bar{L}_k]) \right) e^{h_\varepsilon} + \sum_{j=1}^{n-1} a_j^\varepsilon \partial\rho([L_j, \bar{L}_k]) \\ &= (-dh_\varepsilon(\bar{L}_k) + \alpha(\bar{L}_k)) \frac{e^{h_\varepsilon}}{2} + \sum_{j=1}^{n-1} a_j^\varepsilon \partial\rho([L_j, \bar{L}_k]). \end{aligned} \quad (11)$$

Now let $P \in K$ (i.e. P is a point of infinite type), and let $L_k(P) \in N(P)$. Then $\partial\rho([L_j, \bar{L}_k])(P) = 0$, by pseudoconvexity of $b\Omega$. (A mixed term in a positive semidefinite Hermitian form vanishes if one of the entries is a null direction of the quadratic form.) Consequently

$$\partial\rho([X_\varepsilon, \bar{L}_k])(P) = (-dh_\varepsilon(\bar{L}_k)(P) + \alpha(\bar{L}_k)(P)) \frac{e^{h_\varepsilon(P)}}{2}. \quad (12)$$

Taking into account that both h_ε and α are real, (12) shows that (iii) implies (iv). For the converse implication, fix $P \in K$. We may assume that L_1, \dots, L_{n-1} are orthonormal and that they diagonalize the Levi form at P , and that $L_1(P), \dots, L_m(P)$ span $N(P)$ (for some m with $1 \leq m \leq n-1$). For $m+1 \leq j \leq n-1$, set

$$a_j^\varepsilon := \left(\frac{1}{2} \bar{L}_j h_\varepsilon(P) - \partial\rho([L_n, \bar{L}_j])(P) \right) e^{h_\varepsilon(P)} / \partial\rho([L_j, \bar{L}_j])(P). \quad (13)$$

The field X_ε^p , defined by $X_\varepsilon^p := e^{h_\varepsilon} L_n + \sum_{j=m+1}^{n-1} a_j^\varepsilon L_j$, (where h_ε comes from (iv), and satisfies (5), (6)) then satisfies

$$|\partial\rho([X_\varepsilon^p, \bar{L}_j])(P)| \leq \tilde{C}\varepsilon, \quad 1 \leq j \leq n-1. \quad (14)$$

From here on, the argument is exactly the same as that in the proof of the lemma in [3], pp. 85–86: (14) extends by continuity into a neighborhood of P , and patching finitely many of these locally defined fields via a partition of unity (K is compact) yields a field X_ε defined in a neighborhood of K in $b\Omega$ that has the required commutation properties (i.e. (1) and (2)) with sections of $T^{(0,1)}(b\Omega)$ (note that the function h_ε is defined globally, so that the normal component $e^{h_\varepsilon}L_n$ is defined globally; consequently, terms in the commutators arising from derivatives of the cut-off functions are complex tangential and vanish when $\partial\rho$ is applied); the field obtained in this way can be corrected by a field identically zero on $b\Omega$ to accommodate commutators with \bar{L}_n .

To see that (i) implies (iii), let $\rho_\varepsilon = e^{h_\varepsilon}\rho$ (thus defining h_ε), $P \in K$, and $\bar{L}_k(P) \in T_{b\Omega}^{0,1}(P)$. Then

$$\begin{aligned} e^{h_\varepsilon(P)}\partial\rho([e^{-h_\varepsilon}L_n, \bar{L}_k])(P) &= \sum_j \bar{L}_k \left(e^{-h_\varepsilon} \frac{2}{|\nabla\rho|^2} \frac{\partial\rho}{\partial\bar{z}_j} \right) (P) \frac{\partial\rho}{\partial z_j}(P) e^{h_\varepsilon(P)} \\ &= - \sum_j e^{-h_\varepsilon(P)} \frac{2}{|\nabla\rho(P)|^2} \frac{\partial\rho}{\partial\bar{z}_j}(P) \bar{L}_k \left(\frac{\partial\rho}{\partial z_j} e^{h_\varepsilon} \right) (P), \\ &= -e^{-h_\varepsilon} L_{\rho_\varepsilon}(L_n, \bar{L}_k)(P), \end{aligned} \tag{15}$$

where L_{ρ_ε} denotes the complex Hessian of ρ_ε . We have used in (15) that $\sum_j \frac{2}{|\nabla\rho|^2} \frac{\partial\rho}{\partial\bar{z}_j} \frac{\partial\rho}{\partial z_j} \equiv 1/2$, that $\frac{\partial\rho}{\partial z_j} e^{h_\varepsilon} = \frac{\partial}{\partial z_j}(\rho e^{h_\varepsilon})$ on $b\Omega$, and that \bar{L}_k is tangential. If now $L_k(P) \in N(P)$, we obtain from (8) by polarization (in view of the uniform bounds on $|\nabla\rho_\varepsilon| \approx e^{h_\varepsilon}$ given by (7))

$$|\partial\rho([e^{-h_\varepsilon}L_n, \bar{L}_k])(P)| = O(\varepsilon). \tag{16}$$

From here on, the construction of the family of vector fields required in (iii) proceeds as in the proof above of the implication (iv) \Rightarrow (iii) (which is, as we pointed out, as in [3]).

Conversely, if (iii) is satisfied, then, setting $X_\varepsilon = e^{h_\varepsilon}L_n +$ complex tangential terms, we have (as in (11), (12)),

$$\partial\rho([X_\varepsilon, \bar{L}_k])(P) = \partial\rho([e^{h_\varepsilon}L_n, \bar{L}_k])(P) \tag{17}$$

for $L_k(P) \in N(P)$. (15) gives (replacing h_ε by $-h_\varepsilon$) for $\rho_\varepsilon := e^{-h_\varepsilon}\rho$

$$|L_{\rho_\varepsilon}(L_n, \bar{L}_k)(P)| = O(\varepsilon). \tag{18}$$

By the discussion immediately preceding the definition of a family of essentially pluriharmonic defining functions, (18) suffices to obtain such a family.

(15) also shows that (ii) \Rightarrow (iii) (again using that commutators in directions not in the null space of the Levi form can be adjusted by adding suitable complex tangential terms, as in the proof that (iv) \Rightarrow (iii)).

Going in the other direction, we have with $X_\varepsilon = e^{-h_\varepsilon} L_n +$ complex tangential terms, and $\bar{L}_k(P) \in N(P)$

$$\begin{aligned} \partial\rho([X_\varepsilon, \bar{L}_k])(P) &= \partial\rho([e^{-h_\varepsilon} L_n, \bar{L}_k])(P) \\ &= O(\varepsilon). \end{aligned} \tag{19}$$

Combining (19) with (15) gives, if we set $N_\varepsilon := e^{h_\varepsilon} \sum_{j=1}^n \frac{\partial\rho}{\partial\bar{z}_j} \frac{\partial}{\partial z_j}$, that $\bar{L}_k(\bar{N}_\varepsilon)(P)$ (where \bar{L}_k acts componentwise as in (4)) has inner product with $\bar{L}_n(P)$ that is $O(\varepsilon)$. The inner products with $\bar{L}_1(P), \dots, \bar{L}_{n-1}(P)$ are zero solely by virtue of the fact that $L_k(P) \in N(P)$, regardless of h_ε . Indeed, fix $j \in \{1, \dots, n-1\}$ and let $L_j(P) = (\zeta_1, \dots, \zeta_n)$, $L_k(P) = (w_1, \dots, w_n)$. Then

$$\begin{aligned} \langle \bar{L}_k(\bar{N}_\varepsilon)(P), \bar{L}_j(P) \rangle &= \sum_{j,k} \zeta_j \bar{w}_k \frac{\partial}{\partial \bar{z}_k} \left(e^{h_\varepsilon} \frac{\partial\rho}{\partial z_j} \right) (P) \\ &= e^{h_\varepsilon} \sum_{j,k} \frac{\partial^2 \rho(P)}{\partial z_j \partial \bar{z}_k} \zeta_j \bar{w}_k \\ &= e^{h_\varepsilon} L_\rho(L_j, \bar{L}_k)(P) = 0. \end{aligned} \tag{20}$$

We have used in the second equality that $\sum_j \frac{\partial\rho}{\partial z_j}(P) \zeta_j = 0$, and that Ω is pseudoconvex and $L_k(P) \in N(P)$ in the last equality. Because $\{\bar{L}_1(P), \bar{L}_2(P), \dots, \bar{L}_{n-1}(P), \bar{L}_n(P)\}$ is a basis (over \mathbb{C}) of \mathbb{C}^n , we obtain that $\bar{L}_k(\bar{N}_\varepsilon)(P)$ is $O(\varepsilon)$.

This concludes the proof of the theorem.

Remark 2. As noted in the introduction, the existence of a family of conjugate normals with suitable holomorphicity properties can lead to the existence of Stein neighborhood bases and Mergelyan type approximation theorems. This follows from [1]. We briefly discuss one such instance here; it arises from our work in [15].

We consider smoothly bounded pseudoconvex domains in \mathbb{C}^2 whose set K of infinite type boundary points has a smooth boundary, as a subset of $b\Omega$. The interior of K is then foliated by 1-dimensional complex manifolds. This foliation is usually referred to as the Levi foliation of $\overset{\circ}{K}$. For generic such K , the conditions in the theorem above are satisfied; in fact the conjugate normals can be taken to be holomorphic along the leaves of the Levi foliation of K , that is, they are CR -functions on K .

Γ , the boundary of K , is a 2-dimensional surface sitting inside $b\Omega$, and complex tangents occur precisely at points of Γ where the tangent space of Γ coincides with the complex tangent space to $b\Omega$. Recall that a generic complex tangency is one that is either elliptic or hyperbolic. (This goes back to Bishop's paper [2]; see the introduction of [12] for a thorough discussion of these matters).

For terminology from foliation theory, in particular for the notion of (infinitesimal) holonomy, the reader may consult [6].

Combining our ideas from [15] with ideas from [1] yields the following result, which may be viewed, in the case of \mathbb{C}^2 , as a strengthening of some aspects of Theorem 6.3 of [1] (their theorem addresses the situation in \mathbb{C}^n for general n , however).

Proposition. *a) Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^2 . Suppose that the set K of weakly pseudoconvex boundary points is smoothly bounded (in $b\Omega$) and that its boundary Γ has only isolated generic complex tangencies. Assume that the two leaves that meet at a hyperbolic point of Γ have no other hyperbolic points in their closure (in K). If each leaf of the Levi foliation is closed (in $\overset{\circ}{K}$) and has trivial infinitesimal holonomy, then there exists a conjugate normal field which is CR on K .*

b) If in addition K is uniformly H -convex, then there exists a holomorphic vector field in a neighborhood of K (in \mathbb{C}^n) that is transverse to $b\Omega$ near K .

It is part of the assumption in a) that the two local leaves that meet at a hyperbolic point are globally distinct; see the discussion in [15]. Note that if the leaves of the Levi foliation are assumed simply connected (i.e. they are the analytic discs), then the (infinitesimal) holonomy is automatically trivial. Recall that K is uniformly H -convex if it admits a Stein neighborhood basis $\{U_j\}_{j=1}^\infty$ of open pseudoconvex sets such that for some constant $c > 0$, $\{z \mid \text{dist}(z, K) < \frac{1}{c_j}\} \subseteq U_j \subseteq \{z \mid \text{dist}(z, K) < \frac{c}{j}\}$.

Corollary. *Under the assumptions of the proposition, part b), we have*

a) $\overline{\Omega}$ admits a Stein neighborhood basis

b) Functions analytic in Ω and continuous on $\overline{\Omega}$ can be approximated uniformly on $\overline{\Omega}$ by functions holomorphic in some neighborhood of $\overline{\Omega}$.

The corollary follows directly from the proposition, part b), and [1], §7, in particular Lemma 7.3 (for a)); and [11], Theorem 1 (for b)).

To prove the proposition, we note that part a) comes from [15], Theorem 1, and the theorem above: Theorem 1 in [15] gives a family of vector fields in a neighborhood of K that commute approximately with $\bar{\partial}$, and our theorem above then gives a family of conjugate normals which are approximately holomorphic in weakly pseudoconvex directions (i.e. along the leaves of the Levi foliation of $\overset{\circ}{K}$). Inspection of the proofs (both in [15] and in the theorem above) shows that actually these conjugate normal fields can be taken to be exactly holomorphic on the leaves, i.e. CR on K .

The proof of b) now follows entirely by the arguments in [1]. If K is uniformly H -convex, CR -functions in K can be approximated uniformly on K by functions holomorphic in a neighborhood of K (see e.g. [1], proof of Proposition 6.2). Denoting the algebra of these functions by $A(K)$, we may furthermore invoke Theorem 2.12 in [13] to conclude that the maximal ideal space of $A(K)$

coincides with K . (Note that the assumption that K is uniformly H -convex, needed in the above approximation argument, also guarantees that K is a so-called S_δ in Rossi's terminology, i.e. the intersection of a sequence of pseudoconvex domains.) If we denote the CR conjugate normal by (g_1, \dots, g_n) , then by the above approximation result, $g_j \in A(K)$, $1 \leq j \leq n$. Because the g_j have no common zeros on K (see (3) above), Rossi's result implies that the ideal they generate is all of $A(K)$ (since they are not contained in any maximal ideal). In particular, there exist $f_1, \dots, f_n \in A(K)$ such that $\sum_{j=1}^n f_j g_j \equiv 1$ on K . Since (g_1, \dots, g_n) is *conjugate* normal, this says that the (complex) inner product of (f_1, \dots, f_n) with the normal $(\bar{g}_1, \dots, \bar{g}_n)$ is identically equal to 1, hence so is its real part. Consequently, (f_1, \dots, f_n) is transverse to $b\Omega$ on K . Approximation of f_1, \dots, f_n by functions holomorphic in a neighborhood of K gives a holomorphic vector field near K that is transverse to $b\Omega$. This proves b) and completes the proof of the proposition.

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